

# All Exact Solutions of Non-Abelian Vortices from Yang-Mills Instantons

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## Abstract

We successfully exhaust the complete set of exact solutions of non-Abelian vortices in a quiver gauge theory, that is, the  $S[U(N) \times U(N)]$  gauge theory with a bi-fundamental scalar field on a hyperbolic plane with a certain curvature, from  $SO(3)$ -invariant  $SU(2N)$  Yang-Mills instanton solutions. This work provides, for the first time, exact non-Abelian vortex solutions. We establish the ADHM construction for non-Abelian vortices and identify all the moduli parameters and the complete moduli space.

## I. INTRODUCTION

Since the discovery of non-Abelian vortices [1–3], they have been studied extensively [4–6]. While they are a natural extension of Abelian vortices [7, 8] appearing in conventional superconductors, their analogues also appear in high-density quantum chromodynamics (QCD) showing color superconductors [9]. In supersymmetric gauge theories, they are Bogomol’nyi-Prasad-Sommerfield (BPS) solitons [10] and are stable not only classically but also perturbatively and non-perturbatively. BPS non-Abelian vortices serve as an elegant tool to demonstrate [11] the coincidence of BPS spectra in four-dimensional gauge theories and two-dimensional sigma models [12]. Non-Abelian vortices also play prominent roles as instantons in non-perturbative dynamics of gauge theories in lower dimensions, similar to the role of Yang-Mills instantons [13] in four dimensions; the non-perturbative partition function has been extensively studied by the vortex counting in  $\mathcal{N} = (2, 2)$  supersymmetric gauge theories in two dimensions [14], similar to the instanton counting in four dimensions [15].

However, vortex equations are not integrable even in the BPS limit [16], and explicit solutions and the moduli space metric are not available. This is in contrast to the case of the self-dual equations for Yang-Mills instantons, for which the well-known Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is available [17]. Thus far, some efforts to obtain the moduli space have been made. The moduli space of non-Abelian vortices was determined without the moduli space metric by solving half of the BPS equations [3]. The moduli space metric was obtained implicitly with a matrix function satisfying a differential equation [18]. The asymptotic metric for well-separated vortices was obtained [19] to study the low-energy scattering [20]. The metrics were also obtained on submanifolds for the coincidence limit [21] and on the symmetry orbits [22]. However, the full moduli space is far beyond our reach because of the non-integrability.

Nevertheless, with changes to the geometry, the situation can become totally different. The BPS Abelian vortex equations on the hyperbolic plane  $\mathbb{H}^2$  of curvature  $-1/2$  are integrable [23], and a general formula for the exact moduli space metric has been obtained [24]. The integrability is a consequence of the fact that these vortices are obtained as a dimensional reduction from  $SO(3)$ -symmetric Yang-Mills instantons on flat space  $\mathbb{R}^4$  [23]. More generally, the vortex equations on Riemann surfaces  $\Sigma$  are integrable, when they are

obtained from self-dual Yang-Mills equations on  $\Sigma \times S^2$  [25]. Recently, hyperbolic vortices have been studied extensively [26]. In particular, BPS non-Abelian vortex equations in a quiver gauge theory, *i.e.*,  $S[U(N) \times U(N)]$  gauge theory coupled with a bi-fundamental scalar field on a hyperbolic space were obtained from  $SO(3)$ -symmetric  $SU(2N)$  Yang-Mills instantons in a previous study [27]. However, that study only considered embedding of the Abelian vortex solutions into the diagonal  $U(1)^N$  subgroup. The same vortices on a flat space were also studied in another work [28]. However, there remain open question on whether these vortices have non-trivial orientational moduli or what the complete set of solutions is.

In this Letter, we construct, for the first time, a complete set of all the exact solutions of non-Abelian vortices in the  $S[U(N) \times U(N)]$  gauge theory with a bi-fundamental scalar field on a hyperbolic plane with a certain curvature. We use  $SO(3)$ -invariant  $SU(2N)$  Yang-Mills instanton solutions. We also establish the ADHM construction for non-Abelian vortices and identify all the moduli parameters and the complete moduli space.

## II. HYPERBOLIC VORTICES FROM $SO(3)$ -INVARIANT INSTANTONS

### A. $S[U(N) \times U(N)]$ vortices on a hyperbolic plane

We consider a hyperbolic plane  $\mathbb{H}^2$  as the upper half plane with a complex coordinate  $z$ , with  $r \equiv \text{Im}z > 0$ , endowed with the metric

$$g_{z\bar{z}} = -\frac{2R^2}{(z - \bar{z})^2} = \frac{R^2}{2r^2}. \quad (1)$$

The constant  $R$  is related to the scalar curvature  $-1/R^2$ . See Appendix A.

Let us consider the  $U(N) \times U(N)$  gauge theory with gauge fields  $A_z(z, \bar{z})$  and  $\tilde{A}_{\bar{z}}(z, \bar{z})$ , coupled with a single bi-fundamental Higgs field  $H(z, \bar{z})$ . Note that the overall  $U(1)$  gauge group is trivial, and hence, the actual gauge group is  $S[U(N) \times U(N)]$ . For simplicity, we take the gauge coupling  $g$  to be common for all gauge groups. The covariant derivative is  $\mathcal{D}_{\bar{z}}H = (\partial_{\bar{z}} + iA_{\bar{z}}H - iH\tilde{A}_{\bar{z}})$ . In this setup, the energy of our model is expressed as

$$E = v^2 \int d^2x \, \sigma \, \text{tr} \left[ \frac{1}{\sigma^2} |F_{z\bar{z}}|^2 + \frac{1}{\sigma^2} |\tilde{F}_{\bar{z}z}|^2 + \frac{2}{\sigma} |\mathcal{D}_z H|^2 + \frac{2}{\sigma} |\mathcal{D}_{\bar{z}} H|^2 + \frac{\lambda}{4} (HH^\dagger - \mathbf{1}_N)^2 + \frac{\lambda}{4} (H^\dagger H - \mathbf{1}_N)^2 \right], \quad (2)$$

where the Higgs field  $H$  is rescaled so that its vacuum expectation value  $v$  becomes the overall constant. The function  $\sigma$  is the rescaled hyperbolic metric defined by

$$\sigma \equiv \frac{g^2 v^2}{2} g_{z\bar{z}}. \quad (3)$$

In this Letter, we consider the critical coupling (the BPS limit)  $\frac{\lambda}{4} = 1$  and the “integrable” case:

$$R = \frac{1}{gv} \iff \sigma = \frac{1}{4r^2}. \quad (4)$$

The energy in Eq. (2) can be rewritten as the Bogomol’nyi completion:

$$\begin{aligned} E = v^2 \int d^2x \operatorname{tr} & \left[ \sigma \left| i\sigma^{-1} F_{z\bar{z}} + HH^\dagger - \mathbf{1}_N \right|^2 + \sigma \left| i\sigma^{-1} \tilde{F}_{z\bar{z}} - H^\dagger H + \mathbf{1}_N \right|^2 + 4|\mathcal{D}_{\bar{z}}H|^2 \right. \\ & \left. + 2\mathcal{D}_{\bar{z}}(\mathcal{D}_z HH^\dagger) - 2\mathcal{D}_z(\mathcal{D}_{\bar{z}} HH^\dagger) + 2i(F_{z\bar{z}} - \tilde{F}_{z\bar{z}}) \right]. \end{aligned} \quad (5)$$

The BPS vortex equations can be thus given as follows

$$0 = \mathcal{D}_{\bar{z}}H, \quad (6)$$

$$0 = i\sigma^{-1}F_{z\bar{z}} + HH^\dagger - \mathbf{1}_N, \quad (7)$$

$$0 = i\sigma^{-1}\tilde{F}_{z\bar{z}} - H^\dagger H + \mathbf{1}_N. \quad (8)$$

## B. $SU(2N)$ instanton to $S[U(N) \times U(N)]$ vortices

Here, we consider the  $SO(3)$ -rotationally-invariant  $SU(2N)$  Yang-Mills instantons in four-dimensional Euclidean space  $\mathbb{R}^4$ . The  $SO(3)$  action in space  $(x^1, x^2, x^3)$  leaves the  $x^4$ -axis as a fixed line, while the  $SO(3)$  orbits are  $S^2$ , as shown in Fig. 1. We obtain an upper-half plane  $\mathbb{H}^2$  by an  $S^2$ -dimensional reduction from  $\mathbb{R}^4$  with the  $SO(3)$  fixed line (the  $x^4$ -axis) removed. Since the classical pure Yang-Mills theory is conformally invariant, the conformal equivalence,  $\mathbb{R}^4 - \mathbb{R} \sim \mathbb{H}^2 \times S^2$ , implies that the  $SO(3)$ -invariant  $SU(2)$  instantons are reduced by an  $S^2$ -dimensional reduction to  $U(1)$  Abelian-Higgs vortices on a hyperbolic plane  $\mathbb{H}^2$  with a specific curvature [23]. Here, we extend this relation to the non-Abelian case [27].

First, let us consider the generators of the  $SU(2N)$  gauge group, which are invariant under the diagonal group of the spatial rotation  $SO(3)$  and  $SU(2) \subset SU(2N)$  generated by  $\mathbf{1}_N \otimes \sigma_i$ . It is convenient to take the following basis for the invariant generators

$$\Lambda = T \otimes P, \quad \tilde{\Lambda} = \tilde{T} \otimes \tilde{P}, \quad (9)$$

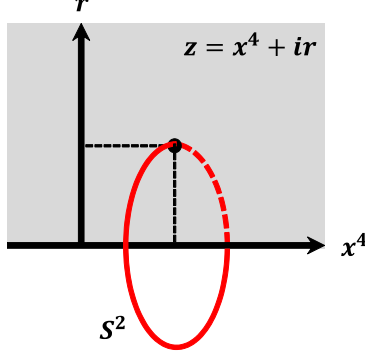


FIG. 1: We consider the  $SO(3)$  action on  $(x^1, x^2, x^3)$ . The  $x^4$  axis is a fixed line. The grey region is an upper-half plane or a hyperbolic surface  $\mathbb{H}^2$ .

where  $T_{\pm}$  are  $N$ -by- $N$  Hermitian matrices that can be viewed as the generators of  $S[U(N) \times U(N)]$ . The 2-by-2 matrices  $P$  and  $\tilde{P}$  are the projection operators defined by

$$P \equiv \frac{\mathbf{1}_2 - \hat{x}_i \sigma_i}{2}, \quad \tilde{P} \equiv \frac{\mathbf{1}_2 + \hat{x}_i \sigma_i}{2}, \quad (10)$$

where  $\sigma_i$  ( $i = 1, 2, 3$ ) denote the Pauli matrices. We define the complex coordinate on  $\mathbb{H}^2$  by  $z \equiv x^4 + ir$  with  $r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ , whereas a unit vector for  $S^2$  is denoted by  $\hat{x}_i \equiv x_i/r$ .

The general  $SU(2)$ -invariant gauge one-form on  $\mathbb{R}^4$  takes the form

$$A_{\mathbb{R}^4} = A \otimes P + \tilde{A} \otimes \tilde{P} - \frac{1}{2}(H - \mathbf{1}_N) \otimes \omega - \frac{1}{2}(H^\dagger - \mathbf{1}_N) \otimes \omega^\dagger, \quad (11)$$

where  $\omega$  is the  $SU(2)$ -invariant one-form on  $S^2$  defined by

$$\omega \equiv iP\sigma_i d\hat{x}_i = i\sigma_i d\hat{x}_i \tilde{P} = \frac{i}{2r}(\delta_{ij} - \hat{x}_i \hat{x}_j - i\epsilon_{ijk} \hat{x}_k) \sigma_j dx_i. \quad (12)$$

As we will see, the one-forms  $A$  and  $\tilde{A}$  can be interpreted as the gauge fields on  $\mathbb{H}^2$ . For the gauge field (11), the field strength  $F_{\mathbb{R}^4} = dA_{\mathbb{R}^4} + iA_{\mathbb{R}^4} \wedge A_{\mathbb{R}^4}$  is given by

$$F_{\mathbb{R}^4} = F \otimes P + \tilde{F} \otimes \tilde{P} + \frac{1}{2}\mathcal{D}H \otimes \omega + \frac{1}{2}\mathcal{D}H^\dagger \otimes \omega^\dagger - \frac{i}{4}(H^\dagger H - \mathbf{1}_N) \otimes \omega^\dagger \wedge \omega - \frac{i}{4}(HH^\dagger - \mathbf{1}_N) \otimes \omega \wedge \omega^\dagger. \quad (13)$$

Substituting this field strength into the Yang-Mills action and integrating over  $S^2$ , we find that the 4d Yang-Mills action reduces to the energy given in Eq. (2) of the 2d  $U(N) \times U(N)$

gauge theory for the integrable case (4)

$$\frac{1}{g_4^2} \int \text{Tr}[F_{\mathbb{R}^4} \wedge *F_{\mathbb{R}^4}] = E, \quad v^2 = \frac{4\pi}{g_4^2}, \quad (14)$$

where we have used the relations  $*(dz \wedge d\bar{z}) \otimes \mathbf{1}_2 = r^2(\omega^\dagger \wedge \omega - \omega \wedge \omega^\dagger)$ ,  $*(dz \wedge \omega) = -dz \wedge \omega$ ,  $*(d\bar{z} \wedge \omega) = d\bar{z} \wedge \omega$ , and  $\int_{S^2} \frac{1}{2i} \text{tr}[\omega \wedge \omega^\dagger] = -4\pi$ . Then, the anti-self-dual equation,  $F_{\mathbb{R}^4} = -*F_{\mathbb{R}^4}$ , for Yang-Mills instantons reduces to the BPS vortex equations (6), (7), and (8) on a hyperbolic plane  $\mathbb{H}^2$  in the integrable case (4). Therefore, we can use  $SU(2N)$  instanton solutions to obtain  $S[U(N) \times U(N)]$  vortex solutions.

### C. $SU(2N)$ instantons from the ADHM construction

In order to construct  $SO(3)$ -invariant instanton solutions, we use the ADHM construction [17]. Let  $B_1$  and  $B_2$  be  $k \times k$  complex matrices and  $I$  and  $J$  be  $k \times 2N$  and  $2N \times k$  complex matrices, respectively. Then, “the zero-dimensional Dirac operator” is defined by

$$\nabla^\dagger = \left( \begin{array}{c|cc} I & z_2 - B_2 & z_1 - B_1 \\ J^\dagger & -(\bar{z}_1 - B_1^\dagger) & \bar{z}_2 - B_2^\dagger \end{array} \right), \quad (15)$$

where we have defined  $z_1 \equiv ix_1 - x_2$ ,  $z_2 \equiv x_4 + ix_3$ . Now, let us consider the following  $SO(3)$  action on the ADHM data  $(B_i, I, J)$

$$\nabla^\dagger \rightarrow g \nabla^\dagger h, \quad g = \mathbf{1}_k \otimes U^\dagger, \quad h = \left( \begin{array}{c|c} \mathbf{1}_N \otimes U & \\ \hline & \mathbf{1}_k \otimes U \end{array} \right), \quad (16)$$

where  $U$  is an arbitrary  $SU(2)$  matrix. The matrices  $(B_i, I, J)$  are invariant under the  $SO(3)$  transformation if they take the following forms.

$$B_1 = B_1^\dagger = 0, \quad B_2 = B_2^\dagger = T, \quad \left( \begin{array}{c} I \\ J^\dagger \end{array} \right) = \left( \begin{array}{c|c} \psi & 0 \\ \hline 0 & \psi \end{array} \right) = \psi \otimes \mathbf{1}_2, \quad (17)$$

where  $T$  is an arbitrary  $k$ -by- $k$  Hermitian matrix and  $\psi$  is an arbitrary  $k$ -by- $N$  matrix. We can show that the  $SO(3)$ -invariant ADHM data automatically satisfy the following ADHM equations.

$$0 = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad 0 = [B_1, B_2] + IJ. \quad (18)$$

More generally, the ADHM equations are satisfied if  $B_1$  and  $B_2$  are diagonal and  $(I, J^\dagger)$  take the form (17). In such a case, the operator  $\nabla^\dagger$  is given by

$$\nabla^\dagger = \begin{pmatrix} \psi \otimes \mathbf{1}_2 & (x^\mu - T^\mu) \otimes \bar{e}_\mu \end{pmatrix}, \quad (19)$$

where  $e_\mu = (-i\sigma_i, \mathbf{1}_2)$  and  $\bar{e}_\mu = (i\sigma_i, \mathbf{1}_2)$ , and we have taken  $B_1 = iT_1 - T_2$  and  $B_2 = T_4 + iT_3$  with  $k \times k$  mutually commuting Hermitian matrices  $T_\mu$ . For notational simplicity, first, we deal with the case of the mutually commuting matrices  $T_\mu$  and then return to the  $SO(3)$ -invariant case by setting  $T_4 = T$  and  $T_i = 0$  ( $i = 1, 2, 3$ ).

For the operator  $\nabla^\dagger$  of the form (19), the zero modes  $V$ , which are a  $(2N + 2k) \times 2N$  complex matrix satisfying the equation

$$\nabla^\dagger V = 0, \quad (20)$$

are found to be

$$V = \begin{pmatrix} \mathbf{1}_N \otimes \mathbf{1}_2 \\ -(x^\nu - T^\nu) [(x^\mu - T^\mu)^2]^{-1} \psi \otimes e_\nu \end{pmatrix} (S^{\dagger-1} \otimes \mathbf{1}_2). \quad (21)$$

Here,  $S$  is an  $N$ -by- $N$  matrix determined from the orthogonality condition

$$V^\dagger V = \mathbf{1}_N, \quad (22)$$

or equivalently

$$SS^\dagger = \mathbf{1}_N + \psi^\dagger [(x^\mu - T^\mu)^2]^{-1} \psi, \quad (23)$$

where we have used the identity  $\bar{e}_\mu e_\nu + \bar{e}_\nu e_\mu = 2\delta_{\mu\nu} \mathbf{1}_2$ . From the matrix  $V$ , the instanton solutions can be explicitly given by

$$A_{\mathbb{R}^4} = -iV^\dagger dV = -\frac{i}{2} S^{-1} \partial^\mu S \otimes (\delta_{\mu\nu} \mathbf{1}_2 - i\eta_{\mu\nu}^{(+)}) dx^\nu + (\text{h.c.}), \quad (24)$$

where  $\eta_{\mu\nu}^{(+)}$  is the self-dual 't Hooft tensor defined by  $\eta_{\mu\nu}^{(+)} = \frac{1}{2i} (\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu)$ . This solution can be viewed as a generalization of the 't Hooft's multi-instanton configuration for the  $SU(2)$  gauge group.

For our purpose, we impose the  $SO(3)$  invariance by setting  $T_4 = T$  and  $T_i = 0$  ( $i = 1, 2, 3$ ). In this case, Eq. (23) indicates that matrix  $S$  is independent of the coordinates of  $S^2$ . Thus, the solutions become

$$A_{\mathbb{R}^4} = i \left[ -S^{-1} \partial_{\bar{z}} S \otimes P + \partial_{\bar{z}} S^\dagger S^{\dagger-1} \otimes \tilde{P} \right] d\bar{z} - ir (S^{-1} \partial_z S + \partial_z S^\dagger S^{\dagger-1}) \otimes \omega + (\text{h.c.}), \quad (25)$$

where we have used  $(\delta_{\mu\nu} \mathbf{1}_2 - i\eta_{\mu\nu}^{(+)}) dx^\nu \partial_\mu = (\bar{e}_\mu dx^\mu) e^\nu \partial_\nu = 2(dzP + r\omega) \partial_z + 2(d\bar{z}\tilde{P} - r\omega^\dagger) \partial_{\bar{z}} +$  derivatives on  $S^2$ .

### III. ALL EXACT $S[U(N) \times U(N)]$ VORTEX SOLUTIONS

#### A. Exact solutions

Comparing Eq. (11) with Eq. (25), we can obtain the vortex solutions  $A_z$  and  $H$ . Let  $T$  be a  $k \times k$  Hermitian matrix and  $\psi$  be a  $k \times N$  complex matrix, made of the respective moduli parameters. The general form of the vortex solution is

$$A_\alpha = -iW^\dagger \partial_\alpha W, \quad \tilde{A}_\alpha = -i\tilde{W}^\dagger \partial_\alpha \tilde{W}, \quad H = W^\dagger \tilde{W}, \quad (\alpha = z, \bar{z}), \quad (26)$$

where  $W$  and  $\tilde{W}$  are  $(N+k) \times k$  matrices, given by

$$W \equiv \begin{pmatrix} \mathbf{1}_N \\ (\bar{z} - T)^{-1} \psi \end{pmatrix} S^{\dagger-1}, \quad \tilde{W} \equiv \begin{pmatrix} \mathbf{1}_N \\ (z - T)^{-1} \psi \end{pmatrix} S^{\dagger-1} \quad (27)$$

with  $S(z, \bar{z})$  satisfying

$$SS^\dagger = \mathbf{1}_N + \psi^\dagger [(z - T)(\bar{z} - T)]^{-1} \psi. \quad (28)$$

Here,  $S$  is the same matrix as the one for instantons in Eq. (21); condition (28) originates from Eq. (23) with the identification  $z = x^4 + ir$  and  $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ .

#### B. The ADHM construction for vortices

From the fact that the  $(N+k) \times k$  matrices  $W$  and  $\tilde{W}$  in Eq. (27) are analogous to the  $2(N+k) \times 2k$  matrix  $V$  in Eq. (21) for the ADHM construction of instantons, the solutions can be recast into the ADHM form. In fact, for a given ADHM data  $(T, \psi)$ , the matrices  $W$  and  $\tilde{W}$  are solution of the ‘‘Dirac equations’’

$$\nabla_v^\dagger W = 0, \quad \tilde{\nabla}_v^\dagger \tilde{W} = 0, \quad (29)$$

where the ‘‘Dirac operators’’ for the vortices are given by

$$\nabla_v^\dagger \equiv \begin{pmatrix} \psi & T - \bar{z} \end{pmatrix}, \quad \tilde{\nabla}_v^\dagger \equiv \begin{pmatrix} \psi & T - z \end{pmatrix}. \quad (30)$$

These Dirac operators are analogous to those of instantons in Eq. (20). Condition (??) is equivalent to the orthogonality conditions for matrices  $W$  and  $\tilde{W}$ :

$$W^\dagger W = \mathbf{1}_N, \quad \tilde{W}^\dagger \tilde{W} = \mathbf{1}_N. \quad (31)$$

These conditions are also counterparts of Eq. (22) for instantons.



### C. Moduli space

Here, we discuss the moduli parameters encoded in solutions (26), (27), and (28) and identify the moduli space. The solutions have the following redundancy in the moduli data  $(T, \psi)$ :

$$T \rightarrow UTU^{-1}, \quad \psi \rightarrow U\psi, \quad U \in U(k). \quad (32)$$

They can be fixed as

$$T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_k \end{pmatrix}, \quad \psi = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & & \ddots & \vdots \\ \rho_{k1} & \rho_{k2} & \cdots & \rho_{kN} \end{pmatrix}, \quad t_i \in \mathbb{R}, \quad \rho_{ij} \in \mathbb{C}. \quad (33)$$

The remaining  $U(1)^k$  redundancy,  $\rho_{ij} \rightarrow \exp(i\theta_i)\rho_{ij}$ , can be fixed as  $\rho_{ii} \in \mathbb{R}$ . Therefore, the dimension of the  $S[U(N) \times U(N)]$  vortex moduli space is 1/4 of that of the (framed)  $SU(2N)$  instanton moduli space:

$$\dim_{\mathbb{R}} \mathcal{M}_{k,N}^{\text{vortex}} = 2kN. \quad (34)$$

If we further divide the moduli space by the  $SU(N)$  (global) gauge symmetry  $\psi \rightarrow \psi g$ ,  $g \in SU(N)$ , the dimension of the moduli space becomes

$$\dim_{\mathbb{R}}[\mathcal{M}_{k,N}^{\text{vortex}}/SU(N)] = \begin{cases} 2kN - N^2 + 1 & \text{for } k > N \\ k^2 + 1 & \text{for } k \leq N \end{cases}. \quad (35)$$

To determine the physical meaning of the moduli parameters, let us calculate

$$\begin{aligned} \det H &\propto \det (\mathbf{1}_N + \psi^\dagger (z\mathbf{1}_k - T)^{-2} \psi) \\ &\propto \det (\mathbf{1}_k + (z\mathbf{1}_k - T)^{-2} \psi \psi^\dagger) \\ &\propto \det ((z\mathbf{1}_k - T)^2 + \psi \psi^\dagger). \end{aligned} \quad (36)$$

Now, let us define a  $k \times k$  complex matrix  $Z$  by

$$Z \equiv T + iR \quad (37)$$

with a  $k \times k$  Hermitian matrix  $R$  satisfying

$$i[T, R] + \psi \psi^\dagger = R^2. \quad (38)$$

Hence, Eq. (36) can be rewritten as

$$\det H \propto \det(z - Z)(z - Z^\dagger). \quad (39)$$

Since the unbroken gauge symmetry becomes larger inside the vortex cores, the zeros of  $\det H$  can be interpreted as the vortex positions. Therefore, Eq. (39) implies that the eigenvalues of  $Z$  are the vortex positions. It is interesting to see that Eq. (38) is a remnant of the D-term condition of a Kähler quotient construction of the vortex moduli space. We thus obtain

$$\mathcal{M}_{k,N} \simeq \left\{ (Z, \psi) \left| \frac{1}{2}[Z^\dagger, Z] + \psi\psi^\dagger = R^2 \right. \right\} / U(k), \quad (40)$$

where the  $U(k)$  action is  $Z \rightarrow UZU^{-1}$ ,  $\psi \rightarrow U\psi$ , and  $R \rightarrow URU^{-1}$ , as in Eq. (32). This is analogous to the Kähler quotient for  $U(N)$  vortices on flat space  $\mathbb{C}$  [1]. The moduli  $Z$  represent the vortex positions, and the moduli  $\psi$  can be identified as the orientational moduli, which are  $k$  copies of  $\mathbb{C}P^{N-1}$  for the separated vortices. The difference between our hyperbolic case and the flat case is that the right hand side of the D-term condition is  $R^2$  in our case while it is just  $(4\pi/g^2)\mathbf{1}_k$  for the flat case. Since the eigenvalues of  $R$  are the vortex positions in the  $r$  coordinate, our D-term condition can be interpreted as a result of a position-dependent gauge coupling, as can be inferred from Eqs. (2) and (3).

Although the  $\mathbb{C}P^{N-1}$  orientational moduli for a single vortex can be absorbed by a global gauge transformation, the relative orientations change the physical quantities. Let us consider a coincident vortex configuration in the  $N = k = 2$  case. If we set the matrices  $T$  and  $\psi$  as

$$T = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \psi = \begin{pmatrix} \sqrt{r_0^2 + a^2} & 0 \\ 2a & \sqrt{r_0^2 - 3a^2} \end{pmatrix}, \quad a \in \left[0, \frac{r_0}{\sqrt{3}}\right], \quad (41)$$

the matrix  $R$  is solved as

$$R = \begin{pmatrix} r_0 & \frac{r_0 + ia}{\sqrt{r_0^2 + a^2}}a \\ \frac{r_0 - ia}{\sqrt{r_0^2 + a^2}}a & r_0 \end{pmatrix}. \quad (42)$$

In this setting, the matrix  $Z = T + iR$  has the degenerate eigenvalue  $ir_0$ ; hence, the two vortices are coincident. Since the vortex position is independent of  $a$ ,  $a$  parameterizes the internal orientation of the vortices. Indeed, we can see from Eq. (41) that  $(T, \psi)$  reduce to

two copies of the data for an Abelian vortex at  $a = 0$  while  $(T, \psi)$  become identical to the data of two vortices in the Abelian case at  $a = r_0/\sqrt{3}$ . We can confirm that the parameter  $a$  is physical by observing the trace of the magnetic flux  $F_{z\bar{z}} = -\tilde{F}_{z\bar{z}}$ .

$$i\sigma^{-1}\text{tr } F_{z\bar{z}} = 8r^2(a^2 + r_0^2) \left[ \frac{1}{\{|z|^2 + r_0^2 + a(z - \bar{z})\}^2} + \frac{1}{\{|z|^2 + r_0^2 - a(z - \bar{z})\}^2} \right]. \quad (43)$$

#### IV. SUMMARY AND DISCUSSION

In summary, we have constructed a complete set of all the exact solutions of non-Abelian vortices in the  $S[U(N) \times U(N)]$  gauge theory with a bi-fundamental scalar field on a hyperbolic plane with a certain curvature, by using  $SO(3)$ -invariant  $SU(2N)$  Yang-Mills instanton solutions. We also have established the ADHM construction for non-Abelian vortices. We further identified the complete moduli space of  $k$  vortices, consisting of the moduli parameters encoded in the  $k \times k$  matrix  $Z$  for the position moduli and the  $k \times N$  matrix  $\psi$  for the orientational moduli. We have found the Kähler quotient for the moduli space, whose complex dimension is  $kN$  as in the flat case.

Future works on related topics will include studies on the index theorem of vortices in quiver gauge theories, the explicit moduli space metric, and low-energy dynamics of vortices; and an extension to arbitrary gauge groups [29], particularly  $SO(N)$  and  $USp(2N)$  [30], from the Yang-Mills instantons with corresponding groups. Since the hyperbolic surface is topologically equivalent to the flat space, quantum dynamics such as the vortex counting should be studied on the hyperbolic surface.

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## Appendix A: Hyperbolic plane

The hyperbolic plane is a subspace in  $\mathbb{R}^{2,1}$  given by

$$X_1^2 + X_2^2 - X_3^2 = -R^2. \quad (\text{A1})$$

The solution is parameterized by  $\phi \in [0, 2\pi), \rho \in \mathbb{R}_{\geq 0}$  as

$$X_1 = R \cos \phi \sinh \rho, \quad X_2 = R \sin \phi \sinh \rho, \quad X_3 = R \cosh \rho. \quad (\text{A2})$$

The metric is given by

$$ds^2 = dX_1^2 + dX_2^2 - dX_3^2 = R^2(d\rho^2 + \sinh^2 \rho d\phi^2). \quad (\text{A3})$$

which gives a constant scalar curvature  $-1/R^2$ . The hyperbolic plane can be parametrized by a complex coordinate  $y$  in an unit disc as

$$y = \tanh \frac{\rho}{2} e^{i\phi}, \quad |y| < 1. \quad (\text{A4})$$

Then, the metric becomes

$$ds^2 = 2g_{y\bar{y}} dy d\bar{y} = \frac{4R^2}{(1 - |y|^2)^2} dy d\bar{y}. \quad (\text{A5})$$

An upper-half plane is also used to parameterize the hyperbolic plane, where a complex coordinate  $z$  is given by

$$z = \frac{y + i}{1 + iy}, \quad \text{Im } z > 0. \quad (\text{A6})$$

In terms of this coordinate the metric becomes

$$ds^2 = R^2 \frac{dz d\bar{z}}{(\text{Im } z)^2}. \quad (\text{A7})$$

The  $SO(2, 1)$  isometry of  $\mathbb{R}^{2,1}$  acts on  $z$  as

$$z \rightarrow \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}. \quad (\text{A8})$$

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